

# On the Approximate Parametrization Problem of Algebraic Curves

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## Abstract

The problem of parametrizing approximately algebraic curves and surfaces is an active research field, with many implications in practical applications. The problem can be treated locally or globally. We formally state the problem, in its global version for the case of algebraic curves (planar or spatial), and we report on some algorithms approaching it, as well as on the associated error distance analysis.

## Keywords

Rational Curve, Approximate Parametrization, Hausdorff distance

## 1 Introduction

Let us say that, within the development of some algebraic computation, probably coming from an applied problem in geometric modeling or in computer aided geometric design, as for instance the intersection of two implicitly given algebraic surfaces, we get an algebraic (planar or spatial) curve  $\mathcal{D}$  that, because of the nature of the problem we are treating, is expected to be rational. However, because of imprecisions (e.g. in the input data or in the arithmetic used in the process), the curve  $\mathcal{D}$  has positive genus, and hence cannot be parametrized with rational functions. The approximate parametrization problem asks for the computation of an algebraic curve  $\overline{\mathcal{D}}$  of genus zero, being in the vicinity of  $\mathcal{D}$ , as well as a rational parametrization of the curve  $\overline{\mathcal{D}}$ ; since we are dealing with sets, the distance (i.e. the vicinity) between  $\mathcal{D}$  and  $\overline{\mathcal{D}}$  is measured using the Hausdorff distance associated to the usual Euclidean distance in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ; see [1]. We report here on the main ideas developed in [6],[7], [9]. Additional work for this problem can be found in [8], [10], [11]; for the local treatment of the problem, one may check [3], [2], [4], [5].

In the following, we focus on the planar case treatment, developed in [6]. For the space case treatment, we refer to [9]. For this purpose we needed to introduce some new concepts as  $\epsilon$ -points,  $\epsilon$ -genus, etc, where  $\epsilon > 0$  is given. Intuitively speaking, the  $\epsilon$ -singularities are points that, although not singular, are *almost* singular. Additionally, we introduce the notion of  $\epsilon$ -multiplicity. The main difficulty that appears is that, in general, one has more  $\epsilon$ -points than expected. To overtake this difficulty we pass, via an equivalence relation, from the  $\epsilon$ -locus (that is, the union of the  $\epsilon$ -singularities and the exact singularities) to a quotient set with finitely many equivalence classes that we call clusters. These clusters play now the role of the classical singularities. We distinguish two types of clusters: those containing exact non-ordinary singularities and the others. To each cluster we associate a representative and a multiplicity as follows:

- If the cluster contains, at least, one exact non-ordinary singularity we assign as multiplicity the maximum exact multiplicity that the non-ordinary singularities provide through their blowing up, and as representative a non-ordinary singularity in the cluster for which the maximum is achieved the maximum; we store the tuple of singularities generated through the blowing up of the representative.
- If the cluster does not contain exact non-ordinary singularities, we assign as multiplicity the maximum of the  $\epsilon$ -multiplicities of their elements, and as representative an element of the cluster where maximum is achieved.

**Notation.** We use the following terminology.  $\|\cdot\|$  and  $\|\cdot\|_2$  denote the polynomial  $\infty$ -norm and the usual unitary norm in  $\mathbb{C}^2$ , respectively.  $|\cdot|$  denotes the module in the field  $\mathbb{C}$  of complex numbers. The partial derivatives of a polynomial  $g \in \mathbb{C}[x, y]$  are denoted by  $g^{\vec{v}} := \frac{\partial^{i+j} g}{\partial^i x \partial^j y}$  where  $\vec{v} = (i, j) \in \mathbb{N}^2$ ; we assume that  $g^{\vec{0}} = g$ . Moreover, for  $\vec{v} = (i, j) \in \mathbb{N}^2$ ,  $|\vec{v}|_* = i + j$ . Also,  $\vec{e}_1 = (1, 0)$  and  $\vec{e}_2 = (0, 1)$ . In addition, let  $\mathcal{D} \subset \mathbb{C}^2$  be an irreducible plane curve over  $\mathbb{C}$ , and let  $f(x, y)$  be its defining polynomial. Furthermore, let  $\epsilon \in \mathbb{R}$  be such that  $0 < \epsilon < 1$ .

## 2 $\epsilon$ -points

The basic ingredient of our reasoning is the notion of  $\epsilon$ -point; the concept of  $\epsilon$ -point of an algebraic variety was introduced by the authors (see [6], [7], [8]) as a generalization of the notion of approximate root of a univariate polynomial. Let  $P \in \mathbb{C}^2$ , we say that  $P$  is an  $\epsilon$ -(affine) point of  $\mathcal{D}$  if  $|f(P)| < \epsilon\|f\|$ . Moreover, if  $P$  is an  $\epsilon$ -point of  $\mathcal{D}$ , we define the  $\epsilon$ -multiplicity of  $P$  on  $\mathcal{D}$  (we denote it by  $\text{mult}_\epsilon(P, \mathcal{D})$ ) as the smallest natural number  $r \in \mathbb{N}$  satisfying that

- (1)  $\forall \vec{v} \in \mathbb{N}^2$ , such that  $0 \leq |\vec{v}|_* \leq r - 1$ , it holds that  $|f^{\vec{v}}(P)| < \epsilon\|f\|$ ,
- (2)  $\exists \vec{v} \in \mathbb{N}^2$ , with  $|\vec{v}|_* = r$ , such that  $|f^{\vec{v}}(P)| \geq \epsilon\|f\|$ .

In this situation, we say that  $P$  is an  $\epsilon$ -(affine) simple point of  $\mathcal{D}$  if  $\text{mult}_\epsilon(P, \mathcal{D}) = 1$ ; otherwise,  $P$  is an  $\epsilon$ -(affine) singularity of  $\mathcal{D}$ . Furthermore, we say that  $P$  is a  $k$ -pure  $\epsilon$ -singularity of  $\mathcal{D}$ , with  $k \in \{1, 2\}$ , if  $\text{mult}_\epsilon(P, \mathcal{D}) > 1$  and  $|f^{\text{mult}_\epsilon(P, \mathcal{D}) \cdot \vec{e}_k}(P)| \geq \epsilon\|f\|$ . In addition, we say that  $P$  is an  $\epsilon$ -(affine) ramification point of  $\mathcal{D}$  if  $\text{mult}_\epsilon(P, \mathcal{D}) = 1$ , and either  $|f^{\vec{e}_1}(P)| < \epsilon\|f\|$  or  $|f^{\vec{e}_2}(P)| < \epsilon\|f\|$ .

Finally, we introduce the weight of an  $\epsilon$ -singularity. This will be used for defining the  $\epsilon$ -genus. Let  $P$  be an  $\epsilon$ -singularity of  $\mathcal{D}$  and  $r = \text{mult}_\epsilon(P, \mathcal{D})$ . If  $P$  is  $k$ -pure, with  $k \in \{1, 2\}$ , we define the  $k$ -weight of  $P$  as

$$\text{weight}_k(P) = \max_{i=0, \dots, r-1} \left\{ \left| \frac{r! \cdot f^{i \cdot \vec{e}_k}(P)}{i! \cdot f^{r \cdot \vec{e}_k}(P)} \right|^{\frac{1}{r-i}} \right\}.$$

If  $P$  is pure in both directions, we define **weight** of  $P$ , as  $\text{weight}(P) = \max\{\text{weight}_1(P), \text{weight}_2(P)\}$  and as the corresponding  $k$ -weight otherwise.

## 3 $\epsilon$ -rationality

Once we have defined the  $\epsilon$ -singularities and their  $\epsilon$ -multiplicities, we introduce the notion of  $\epsilon$ -genus. This seems easy, since the genus can be introduced by means of multiplicities and we already have the notion of  $\epsilon$ -multiplicity. However, the main problem is that there are more  $\epsilon$ -singularities than expected. To face this problem, we introduce an equivalence relation over the set of  $\epsilon$ -singularities and the equivalence classes would play the role of the  $\epsilon$ -singularities in the  $\epsilon$ -genus formula. More precisely, let  $\mathcal{S}$  be a finite set of  $\epsilon$ -singularities of  $\mathcal{D}$ . In addition, let  $\mathcal{N}$  be the finite set (maybe empty) of exact non-ordinary singularities of  $\mathcal{D}$ . We replace  $\mathcal{S}$  by  $\mathcal{S} \cup \mathcal{N}$ . Also, for  $P \in \mathcal{N}$  we will refer to the tuple of neighboring multiplicities of  $P$ , and we will denote it by  $\text{NeighMult}(P)$ , as the tuple of all exact multiplicities of  $P$  and the neighboring points generated through its blowing up. For  $P \in \mathcal{S}$  we define the radius of  $P$ , and we denote it by  $\text{radius}(P)$ , as

$$\text{radius}(P) = \begin{cases} \mathcal{R}_{\text{out}}(\text{weight}(P)) & \text{if } P \text{ is pure} \\ 0 & \text{otherwise} \end{cases}$$

where  $\mathcal{R}_{\text{out}}$  is Sasaki-Terui out rational function, namely,

$$\mathcal{R}_{\text{out}}(x) = \frac{1}{2} - \frac{x(1-9x)}{2(1+3x)} - \frac{32x^2}{(1+3x)^3}.$$

Now, we introduce the following equivalence relation in  $\mathcal{S}$ . Let  $P, Q \in \mathcal{S}$ , then

$$P \mathcal{R} Q \iff \begin{cases} P \mathcal{R}^* Q \\ \text{or there exist } P_1, \dots, P_n \in \mathcal{S} \text{ such that } P \mathcal{R}^* P_1, \dots, P_n \mathcal{R}^* Q \end{cases}$$

where

$$P \mathcal{R}^* Q \iff \|P - Q\|_2 + |\text{radius}(P) - \text{radius}(Q)| < \mathcal{R}_{\text{out}}(\epsilon).$$

We define the ( $\epsilon$ -singular) clusters of  $\mathcal{D}$  as the equivalence classes in  $\mathcal{S}/\mathcal{R}$ . In addition, we distinguish two type of clusters: those whose intersection with  $\mathcal{N}$  is empty and the others. Let  $\mathfrak{C}^{\text{ord}}$  be the set of all clusters of the first type, and let  $\mathfrak{C}^{\text{non}}$  be the set of all clusters of the second type. So,  $\mathcal{S}/\mathcal{R}$  decomposes as

$$\mathcal{S}/\mathcal{R} = \mathfrak{C}^{\text{ord}} \cup \mathfrak{C}^{\text{non}}.$$

In this situation, if  $\mathfrak{C}^{\text{ord}} = \{\mathcal{C}luster_{r_i}(P_i)\}_{i=1, \dots, s_1}$  and  $\mathfrak{C}^{\text{non}} = \{\mathcal{C}luster_{T_i}(M_i)\}_{i=1, \dots, s_2}$ , with  $T_i = (k_{i,1}, \dots, k_{i,\ell_i})$ , we define the  $\epsilon$ -genus of  $\mathcal{D}$  as

$$\epsilon\text{-genus}(\mathcal{D}) = \frac{(\deg(\mathcal{D}) - 1)(\deg(\mathcal{D}) - 2)}{2} - \sum_{i=1}^{s_1} \frac{r_i(r_i - 1)}{2} - \sum_{i=1}^{s_2} \sum_{j=1}^{\ell_i} \frac{k_{i,j}(k_{i,j} - 1)}{2}.$$

In addition, we say that  $\mathcal{D}$  is  $\epsilon$ -rational if  $\epsilon\text{-genus}(\mathcal{D}) = 0$ .

In [6], for the application of the planar approximate parametrization algorithm, we imposed among other conditions that  $\mathcal{D}$  has proper degree and that  $\mathcal{D}$  is  $\epsilon$ -irreducible over  $\mathbb{C}$ . These two notions depend on  $\epsilon$ . More precisely,  $\mathcal{D}$  has proper degree  $d > 0$  if the total degree of  $f$  is  $\ell$ , and  $\exists \vec{v} \in \mathbb{N}^2$ , with  $|\vec{v}|_* = \ell$ , such that  $|f^{\vec{v}}| > \epsilon \|f\|$ . Moreover, we say that  $\mathcal{D}$  is  $\epsilon$ -irreducible if  $f$  cannot be expressed as  $f(x, y) = g(x, y)h(x, y) + \mathcal{E}(x, y)$  where  $h, g, \mathcal{E} \in \mathbb{F}[x, y]$  and  $\|\mathcal{E}(x, y)\| < \epsilon \|f(x, y)\|$ . Nevertheless we observe that taking, if necessary, a smaller  $\epsilon$  we can avoid the properness requirement on the degree and we can change the  $\epsilon$ -irreducibility of  $\mathcal{D}$  by irreducibility over  $\mathbb{C}$ . Thus, we will ask the planar curve  $\mathcal{D}$  to satisfy the following general conditions:

1.  $\mathcal{D}$  is an affine real plane algebraic curve over  $\mathbb{C}$
2.  $\mathcal{D}$  is irreducible over  $\mathbb{C}$ .
3.  $\mathcal{D}^\infty$  consists in  $d$  different points at infinity, where  $d = \deg(\mathcal{D})$ , note that this, in particular, implies that all points at infinity are regular, and the line at infinity is not tangent to  $\mathcal{D}$ .
4.  $(1 : 0 : 0), (0 : 1 : 0) \notin \mathcal{D}^h$  (where  $\mathcal{D}^h$  denotes the homogenization of  $\mathcal{D}$ ).

Let us mention that the condition  $(1 : 0 : 0), (0 : 1 : 0) \notin \mathcal{D}^h$  can always be achieved by performing a suitable affine orthogonal linear change of coordinates.

In this situation, we have the following algorithm.

**Algorithm:** Given a tolerance  $0 < \epsilon < 1$ , and  $\mathcal{D}$  satisfying the conditions imposed above, the algorithm decides whether  $\mathcal{D}$  is  $\epsilon$ -rational and, in the affirmative case, it computes a rational parametrization  $\overline{\mathcal{P}}(t)$  of a curve  $\overline{\mathcal{D}}$  whose real part is at finite Hausdorff distance of the real part of  $\mathcal{D}$  and such that  $\deg(\mathcal{D}) = \deg(\overline{\mathcal{D}})$ . Let  $f$  be defining polynomial of  $\mathcal{D}$  and  $F$  its homogenization.

- (1) Let  $d = \deg(\mathcal{D})$ . If  $d = 1$  output a polynomial parametrization of the line  $\mathcal{D}$ . If  $d = 2$  apply algorithm from [7] to  $\mathcal{D}$ .
- (2) Compute  $\mathfrak{C}^{\text{ord}} = \{\mathcal{C}luster_{r_i}(Q_i)\}_{i=1, \dots, s_1}$  and  $\mathfrak{C}^{\text{non}} = \{\mathcal{C}luster_{T_i}(M_i)\}_{i=1, \dots, s_2}$  of  $\mathcal{D}$ ; say  $Q_i = (q_{i,1} : q_{i,2} : 1)$ ,  $M_i := (m_{i,1} : m_{i,2} : 1)$  and  $T_i := (k_{i,1}, \dots, k_{i,\ell_i})$ .
- (3) If  $\epsilon\text{-genus}(\mathcal{D}) \neq 0$  RETURN “ $\mathcal{D}$  is not  $\epsilon$ -rational”. If  $s = 1$  one may apply the algorithm from [7] for the monomial case.

- (4) Determine the linear subsystem  $\mathcal{A}_{d-2}$  of adjoints to  $\mathcal{D}$ , of degree  $d-2$ , that has the non-ordinary singularities  $M_i$ , for  $i \in \{1, \dots, s_2\}$ , as base points. Let  $\mathcal{H}_{d-2}$  be the intersection of  $\mathcal{A}_{d-2}$  with the linear system of degree  $(d-2)$  given by the divisor  $\sum_{i=1}^s (r_i - 1)Q_i$ .
- (5) Compute  $(d-3)$   $\epsilon$ -ramification points  $\{P_j\}_{1 \leq j \leq d-3}$  of  $\mathcal{D}$ ; if there are not enough  $\epsilon$ -ramification points, complete with simple  $\epsilon$ -points. Take the points over  $\mathbb{R}$ , or as conjugate complex points. After each point computation check that it is not in the cluster of the others (including the clusters in  $\mathfrak{C}^{\text{ord}} \cup \mathfrak{C}^{\text{non}}$ ); if this fails take a new one. Say  $P_i = (p_{i,1} : p_{i,2} : 1)$ .
- (6) Determine the linear subsystem  $\mathcal{H}_{d-2}^*$  of  $\mathcal{H}_{d-2}$  given by the divisor  $\sum_{i=1}^{d-3} P_i$ . Let  $H^*(t, x, y, z) = H_1(x, y, z) + tH_2(x, y, z)$  be its defining polynomial.
- (7) If  $[\gcd(F(x, y, 0), H_1(x, y, 0)) \neq 1]$  and  $[\gcd(F(x, y, 0), H_2(x, y, 0)) \neq 1]$  replace  $H_2$  by  $H_2 + \rho_1 x^{d-2} + \rho_2 y^{d-2}$ , where  $\rho_1, \rho_2$  are real and strictly smaller than  $\epsilon$ . Say that  $\gcd(F(x, y, 0), H_2(x, y, 0)) = 1$ ; similarly in the other case.
- (8)  $S_1(x, t) = \text{Res}_y(H^*(x, y, 1), f)$  and  $S_2(y, t) = \text{Res}_x(H^*(x, y, 1), f)$ .
- (9)  $A_1 = \prod_{i=1}^{s_1} (x - q_{i,1})^{r_i(r_i-1)} \prod_{i=1}^{s_2} (x - m_{i,1})^{\sum_{j=1}^{\ell_i} k_{i,j}(k_{i,j}-1)} \prod_{i=1}^{d-3} (x - p_{i,1})$ ,  
 $A_2 = \prod_{i=1}^{s_1} (y - q_{i,2})^{r_i(r_i-1)} \prod_{i=1}^{s_2} (y - m_{i,2})^{\sum_{j=1}^{\ell_i} k_{i,j}(k_{i,j}-1)} \prod_{i=1}^{d-3} (y - p_{i,2})$ .
- (10) For  $i = 1, 2$  compute the quotient  $B_i$  of  $S_i$  by  $A_i$  w.r.t. either  $x$  or  $y$ .
- (11) If the content of  $B_1$  w.r.t  $x$  or the content of  $B_2$  w.r.t.  $y$  does depend on  $t$ , RETURN "degenerate case" (see [6]).
- (12) Determine the root  $\bar{p}_1(t)$  of  $B_1$ , as a polynomial in  $x$ , and the root  $\bar{p}_2(t)$  of  $B_2$ , as a polynomial in  $y$ .
- (12) RETURN  $\bar{\mathcal{P}}(t) = (\bar{p}_1(t), \bar{p}_2(t))$ .

In the following Example we illustrate the Algorithm.

**Example 3.1.** Let  $\epsilon = \frac{1}{100}$  and  $\mathcal{D}$  the curve of proper degree 5 defined by (see Fig.3.1):

$$f(x, y) = \frac{8578750}{617673396283947}y^3x^2 - \frac{299200}{7625597484987}yx^3 - \frac{1870000}{617673396283947}y^2x^2 + \frac{56359375}{50031545098999707}y^4x \\ - \frac{11687500}{150094635296999121}y^3x + \frac{17276000}{617673396283947}x^3y^2 - \frac{6055664500}{50031545098999707}x^4y - \frac{47872}{282429536481}x^4 \\ + \frac{1562500}{50031545098999707}y^5 + \frac{3125000}{50031545098999707}x^5.$$

First we compute the  $\epsilon$ -singularities of  $\mathcal{D}$ :

$$\{Q_1 = (0.008215206627 - 0.003422196305I, -0.1256431531 + 0.01292576399I), \\ Q_2 = (0.008215206627 + 0.003422196305I, -0.1256431531 - 0.01292576399I), \\ Q_3 = (0, 0), Q_4 = (0.003676621613, -0.05844533731), Q_5 = (0.02528071675, -0.2879266871)\}.$$

The singularities  $\{Q_1, Q_2\}$  have  $\epsilon$ -multiplicity 3, and  $\{Q_3, Q_4, Q_5\}$  have  $\epsilon$ -multiplicity 4. Moreover, the cluster decomposition of the singular locus consists in an unique cluster taking the maximum  $\epsilon$ -multiplicity 4:  $\mathcal{C}luster_4(Q_3) = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ . And therefore  $\mathcal{D}$  is  $\epsilon$ -rational since it is monomial. Finally, the algorithm outputs the parametrization:

$$\bar{\mathcal{P}}(t) = \left( \frac{748(25t + 324)^3}{375(t - 2)(12500t^4 + 475875t^3 + 6510780t^2 + 24216408t - 12500)}, \right. \\ \left. t \frac{748(25t + 324)^3}{375(t - 2)(12500t^4 + 475875t^3 + 6510780 * t^2 + 24216408t - 12500)} \right).$$

See the following figure to compare the input and the output curves:

**Acknowledgments:** This work was developed, and partially supported, under the research project MTM2011-25816-C02-01. All authors belong to the Research Group ASYNACS (Ref. CCEE2011/R34).

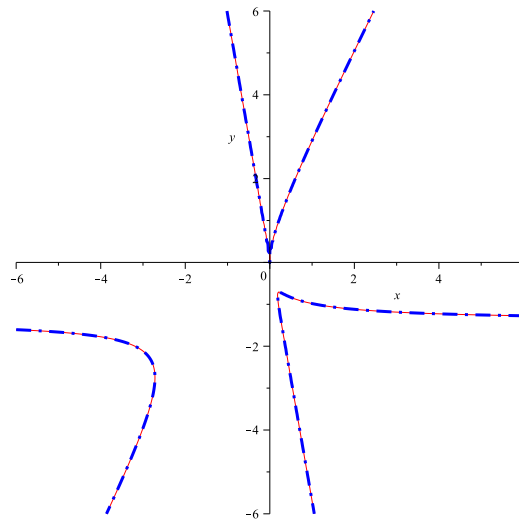


Figure 1: Input (in dots) and output curve.

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